

# Effect of quintessence on the energy of the Reissner-Nordstrom black hole

*Ibrar Hussain and Sajid Ali*

School of Electrical Engineering and Computer Science,  
National University of Sciences and Technology,  
H-12, Islamabad, Pakistan.

E-mail: ibrar.hussain@seecs.nust.edu.pk, sajid.ali@mail.com

**Abstract.** The energy content of the Reissner-Nordstrom black hole surrounded by quintessence is investigated using approximate Lie symmetry methods. It is mainly done by assuming mass and charge of the black hole as small quantities ( $\epsilon$ ), and by retaining its second power in the perturbed geodesic equations for such black hole while neglecting its higher powers. Due to the presence of trivial second-order approximate Lie symmetries of these perturbed geodesic equations, a rescaling of the geodetic parameter gives a rescaling of the energy in this black hole. Interestingly we obtain an explicit relation of the rescaling factor that depends on the square of the charge to mass ratio of the black hole, the normalization factor  $\alpha$ , which is related to the state parameter of the quintessence matter, and the coordinate  $r$ . A comparison of this rescaling factor with that of the Reissner-Nordstrom black hole (Hussain et. al SIGMA, 2007), without quintessence is given. It is observed that the presence of the quintessence field reduces the energy in this black hole spacetime. Further it is found that there exists a point outside the event horizon of this black hole where the effect of quintessence balances the energy content in this black hole without quintessence, and where the total energy of the underlying spacetime becomes zero.

*Key words:* Energy; Reissner-Nordstrom black hole surrounded by quintessence; Approximate Lie symmetries

## 1. Introduction

To give a well accepted definition of energy is a long standing problem in general relativity (GR) since the time of Einstein (for detail one may see [1]). To resolve this problem many scientists have given their own notions of energy in GR [2]. These attempts to define energy in GR include several pseudo-tensors and approximate symmetry approaches. A review of these approaches is available in the literature [3, 4, 5]. The pseudo-tensors are coordinate dependent quantities and therefore violates the basic principle of GR. On the other hand approximate symmetry approaches, except the approximate Lie symme-

try approach have their own drawbacks which are pointed out in [5]. The approximate Lie symmetry method [6], to define energy in GR is comparatively new and free of the drawbacks.

The approximate Lie symmetry methods for differential equations [6], have been applied to the approximate (perturbed) geodesic equations of different gravitational wave spacetimes and some black hole spacetimes of GR and other theories of gravity [5, 7, 8, 9, 10]. For all these spacetimes energy rescaling factors have obtained. These spacetimes include plane-fronted and cylindrical gravitational waves [11, 12], Reissner-Nordstrom [4], Kerr-Newman [13], Kerr-Newman AdS [14], BTZ [7], Bardeen [8], stringy black hole [9] and a slowly rotating black hole in the Horava-Lifshitz theory of gravity [10].

The evidence for the accelerated expansion of our Universe is supported by some Cosmological observations like the Supernovae Ia, the Cosmic Microwave Background radiation anisotropies and X-ray experiments [15, 16, 17]. Astrophysicists and cosmologists consider a missing energy component with negative pressure called dark energy, responsible for this accelerated expansion of the Universe. There are different candidates for dark energy (see e.g., [18]). One of these is known as quintessence. This is defined as a scalar field with the equations of negative state parameter, which is the ratio of the pressure and density [19]. In 2003, Kiselev derived a charged black hole solution with quintessence term, of the Einstein field equations (EFEs) [20]. This solution reduces to the Reissner-Nordstrom black hole solution of the EFEs in the absence of the quintessence term. The energy expression (rescaling factor) in the case of Reissner-Nordstrom black hole was obtained in [4] via approximate Lie symmetry methods. In this paper we are interested to apply the approximate Lie symmetry methods to the Reissner-Nordstrom black hole with quintessence to look at its energy content and compare it with that of the Reissner-Nordstrom black hole. In particular we want to study the effect of the quintessence term in the energy of this black hole.

This paper is organized as follows. In the next Section mathematical definitions to be utilize are given. In Section 3 we will discuss the exact, first-order and the second-order approximate symmetries of the geodesic equations of the perturbed Reissner-Nordstrom black hole surrounded by quintessence. In the same Section we will derive the energy rescaling factor for this black hole. In Section 4 the effect due to the presence of quintessence field is discussed in detail. A summery of the work done here is given in the last Section.

## 2. Basic Definitions

Under a point symmetry transformation [6, 21],

$$\mathbf{V} = \zeta(\tau, x^\alpha) \frac{\partial}{\partial \tau} + \psi^\beta(\tau, x^\alpha) \frac{\partial}{\partial x^\beta}, \quad (1)$$

where  $\alpha, \beta = 0, 1, 2, 3$ , a second-order approximate Lie symmetry is a vector field

$$\mathbf{V} = \mathbf{V}_0 + \epsilon \mathbf{V}_1 + \epsilon^2 \mathbf{V}_2 + O(\epsilon^3), \quad (2)$$

for a system of perturbed ordinary differential equations (here the system of geodesic equations)

$$\mathbf{G} = \mathbf{G}_0 + \epsilon \mathbf{G}_1 + \epsilon^2 \mathbf{G}_2 + O(\epsilon^3), \quad (3)$$

if the following condition holds

$$(\mathbf{V})(\mathbf{G})_{\mathbf{G}=O(\epsilon^3)} = O(\epsilon^3). \quad (4)$$

In (2)  $\mathbf{V}_0$  is the exact (when  $\epsilon = 0$ ) symmetry generator for the system of the exact geodesic equations  $\mathbf{G}_0$ . The exact Lie symmetry can be determined from

$$(\mathbf{V}_0)(\mathbf{G}_0)_{\mathbf{G}=0} = 0. \quad (5)$$

The vector fields  $\mathbf{V}_1$  and  $\mathbf{V}_2$  are the first-order and second-order approximate parts of the approximate symmetry generator  $\mathbf{V}$ , respectively. Similarly  $\mathbf{G}_1$  and  $\mathbf{G}_2$  are the first-order and second-order perturbed parts of the system of geodesic equations  $\mathbf{G}$ , respectively. Since the geodesic equations are second order ordinary differential equations, therefore the second prolongation of the vector fields  $\mathbf{V}_0$ ,  $\mathbf{V}_1$  and  $\mathbf{V}_2$  should be used in (4) and (5), which is available (for example) in [5]. The second-order approximate symmetry generator is said to be nontrivial approximate symmetry generator if it is not proportional to any one of the lower-order (i.e exact or first-order) symmetry generators. The second-order approximate symmetry generator is also called nontrivial if at least one of the lower-order symmetry generators are nonzero for it. In the case of trivial approximate symmetry generators it is also possible that the lower-order symmetry generator cancel out in the set of determining equations which can be obtained from (4). It is worth remarking that the interesting result of energy rescaling (explained in the next Section) comes from the applications of the perturbed system of geodesic equations in the subscript of (4), as required (for more detail see [5]).

The first prolongation of the vector field (1) is

$$\mathbf{V}^{[1]} = \mathbf{V} + (\dot{\psi}_{,\tau}^\alpha + \gamma_{,\beta}^\alpha \dot{x}^\beta - \zeta_{,\tau} \dot{x}^\alpha - \zeta_{,\beta} \dot{x}^\beta \dot{x}^\alpha) \frac{\partial}{\partial \dot{x}^\alpha}. \quad (6)$$

The vector field  $\mathbf{V}$  is called a Noether gauge symmetry generator for the Lagrangian  $L(\tau, x^\beta, \dot{x}^\beta)$ , if the following condition satisfies

$$\mathbf{V}^{[1]}L + (\mathbf{d}\zeta)L = \mathbf{d}g. \quad (7)$$

Here  $g(\tau, x^\beta)$  is a gauge function and the total derivative operator  $\mathbf{d}$  is given by

$$\mathbf{d} = \frac{\partial}{\partial \tau} + \dot{x}^\beta \frac{\partial}{\partial x^\beta}. \quad (8)$$

Throughout this article the dot stands for derivative with respect to the geodesic parameter  $\tau$  and Einstein summation convention is assumed. The following Noether theorem [22], reveals the significance of Noether symmetries.

**Theorem 1.** If  $\mathbf{V}$  is a Noether gauge symmetry generator corresponding to a Lagrangian  $L(\tau, x^\beta, \dot{x}^\beta)$  of the Euler-Lagrange equations of motion, then

$$I = \zeta L + (\psi^\beta - \dot{x}^\beta \zeta) \frac{\partial L}{\partial \dot{x}^\beta} - g, \quad (9)$$

is a constant of motion associated with the symmetry generator  $\mathbf{V}$ . The proof of this theorem can be seen for example in [23].

### 3. Approximate Lie Symmetries and the Energy content of the Reissner-Nordstrom black hole surrounded by quintessence

The line element for the static charged black hole surrounded by quintessence is [24]

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2. \quad (10)$$

where  $d\Omega$  is the solid angle defined by  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$  and  $f(r)$  is a function of the form

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\alpha}{r^{3\omega_q+1}}. \quad (11)$$

In (11)  $M$  is the mass and  $Q$  is the total charge of the black hole. The factor  $\alpha$  is related with the state parameter  $\omega_q$  by

$$\alpha = -\frac{2\rho_q r^{3(\omega_q+1)}}{3\omega_q}, \quad (12)$$

where  $\rho_q$  is the energy density of the quintessence matter. The pressure  $p_q$  for the quintessence matter has the relation with  $\rho_q$  as

$$p_q = \omega_q \rho_q. \quad (13)$$

The range of  $\omega_q$  for quintessence matter is given in the literature (see e.g., [24]). In this paper we take  $\omega_q = -\frac{2}{3}$ , for which the charged black hole with quintessence, given by (10) and (11), becomes the simplest nontrivial case [24]. For this value of  $\omega_q$  the function in (11), takes the following form

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \alpha r, \quad (14)$$

in which case the black hole is non-asymptotically flat.

To discuss the approximate symmetries of the charged-quintessence Reissner-Nordstrom black hole, we assume both of its mass and charge to correspond to first and second order perturbation of  $\epsilon$ , i.e.,

$$2M = \epsilon, \quad Q^2 = k\epsilon^2. \quad (15)$$

To avoid the naked singularity [1], we must have

$$M^2 \geq Q^2, \quad (16)$$

therefor  $0 < k \leq \frac{1}{4}$ . Equation (14) now becomes

$$f(r) = 1 - \alpha r - \frac{\epsilon}{r} + \frac{\epsilon^2}{r^2}. \quad (17)$$

By retaining the second power of  $\epsilon$  and neglecting its higher powers in the perturbed geodesic equation of this spacetime we get

$$\begin{aligned} \ddot{t} &= -\frac{\alpha r \dot{t}}{\alpha r - 1} + \epsilon \frac{(2\alpha r - 1) \dot{r} \dot{t}}{r^2 (\alpha r - 1)^2} - \epsilon^2 \frac{(6k\alpha^2 r^2 - 2\alpha(5k - 2)r + 2(2k - 1)) \dot{r} \dot{t}}{r^3 (\alpha r - 1)^3} + O(\epsilon^3), \\ \ddot{r} &= \frac{\alpha(\dot{r}^2 - (\alpha r - 1)^2 \dot{t}^2)}{2(\alpha r - 1)} + r(1 - \alpha r)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \epsilon \left( \frac{(1 - 2\alpha r) \dot{r}^2 - (\alpha r - 1)^2 \dot{t}^2}{2r^2 (\alpha r - 1)^2} \right. \\ &\quad \left. - (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) + \frac{\epsilon^2}{r^3 (\alpha r - 1)^3} \left( -k\alpha^4 r^4 \dot{t}^2 + (5k + 1)\alpha^3 r^3 \dot{t}^2 + 3\alpha^2 r^2 (k\dot{r}^2 - (3k + 1)\dot{t}^2) + \right. \\ &\quad \left. \alpha r ((7k + 3)\dot{t}^2 - (5k - 2)\dot{r}^2) - (2k + 1)\dot{t}^2 + (2k - 1)\dot{r}^2 + \frac{k}{r}(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right) + O(\epsilon^3), \\ \ddot{\theta} &= \sin \theta \cos \theta \dot{\phi}^2 - \frac{2}{r} \dot{r} \dot{\theta}, \\ \ddot{\phi} &= -\frac{2}{r} \dot{r} \dot{\phi} - 2 \cot(\theta) \dot{\theta} \dot{\phi}, \end{aligned} \quad (18)$$

where the last two equations are the same as for the unperturbed case because there was no perturbation in the solid angle. In order to investigate the second-order approximate symmetries of the above system we first need to obtain the exact and the first-order approximate symmetries of the exact and the first-order perturbed system of geodesic

equations. For the exact case we substitute  $\epsilon = 0$ , in the above system (18) and get six Lie symmetry generators from (5)

$$\begin{aligned} \mathbf{V}_0 &= \tau \frac{\partial}{\partial \tau}, \quad \mathbf{V}_1 = \frac{\partial}{\partial \tau}, \quad \mathbf{V}_2 = \frac{\partial}{\partial t}, \quad \mathbf{V}_3 = \frac{\partial}{\partial \phi}, \\ \mathbf{V}_4 &= \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \quad \mathbf{V}_5 = -\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi}. \end{aligned} \quad (19)$$

In the above symmetry generators  $\mathbf{V}_2$ ,  $\mathbf{V}_3$ ,  $\mathbf{V}_4$ , and  $\mathbf{V}_5$ , are the Killing vectors for the underlying spacetime, which correspond to energy, azimuthal angular momentum and angular momentum conservation. The four Killing vectors together with the symmetry generator  $\mathbf{V}_1$  are the Noether symmetries, i.e., symmetries of the corresponding Lagrangian for the geodesic equations. The corresponding conserved quantities or first integrals of the equations of motion can be obtained by using the Noether theorem given in Section 1. The symmetry generator given by  $\mathbf{V}_0$ , is the proper Lie symmetry generator for the spacetime under consideration that correspond to rescaling transformation of the proper time  $\tau$ .

Using these six exact Lie symmetry generators given in (19), in the definition of first-order approximate Lie symmetry conditions (which can be obtained by retaining only the first power of  $\epsilon$  in (3) and (4)) we obtain the same six symmetry generators which are trivial first-order approximate symmetry generators. Now the second-order approximate symmetry generators can be obtained by retaining the second-order perturbed term  $\epsilon$  and neglecting higher-order terms in the perturbed geodesic equations (18), and by employing the six symmetry generators given in (19) as the exact and the first-order approximate symmetries generators. From the solution of (4), it turns out that we do not get any new symmetry generator for the second-order perturbation and retain those six trivial symmetry generators. Therefore the six exact symmetry generators given in (19) are obtained as trivial second-order approximate symmetries generators.

The trivial second-order approximate symmetry generators  $\mathbf{V}_0$  and  $\mathbf{V}_1$  in (19), correspond to

$$\zeta = a_0 + a_1 \tau, \quad (20)$$

It is observed that in the set of determining equations for the first-order approximate symmetries of the first-order perturbed geodesic equations the terms involving  $\zeta_\tau = a_1$  cancel out. Whereas in the case of second-order approximate symmetries of the second-order perturbed geodesic equations for Reissner-Nordstrom spacetime with quintessence the terms involving  $\zeta_\tau = a_1$ , do not cancel automatically. Interestingly these collect a scaling factor to cancel out for consistency of the determining equations. In which case we get the scaling factor

$$\left[ (1 - 2k) + \alpha r(5k - 2) - 3k\alpha^2 r^2 \right]. \quad (21)$$

The time translation is related with energy conservation and  $\zeta$  is the coefficient of  $\partial/\partial \tau$  in the point symmetry transformation (1), where  $\tau$  is the proper time. The scaling factor

(21) corresponds to the rescaling of energy in the Reissner-Nordstrom spacetime surround by quintessence. By substituting  $k = \frac{Q^2}{4M^2}$ , in (21) we get

$$\left[ \left( 1 - \frac{Q^2}{2M^2} \right) + \alpha r \left( \frac{5Q^2}{4M^2} - 2 \right) - \frac{3Q^2}{4M^2} \alpha^2 r^2 \right]. \quad (22)$$

By putting  $\alpha = 0$ , (22) reduces to the same scaling factor which was obtained for the Reissner-Nordstrom spacetime without quintessence [4]. We see that like the Reissner-Nordstrom spacetime [4], this scaling factor involves the second power of ratio between  $Q$  and  $M$ . Therefore it relates the electromagnetic self-energy to the gravitational self-energy.

## 4. Effect of quintessence

The pressure  $p_q$  of the quintessence field which is assumed to be responsible for the accelerated expansion of the Universe is negative [15]. It is evident from equation (13), that  $\rho_q$  has to be positive because  $\omega_q$  is negative. Therefore, from equation (12) we see that  $\alpha$  is positive. Thus the energy in the field of Reissner-Nordstrom black hole surrounded by quintessence is differ by

$$E_Q(r) = \alpha r(5k - 2) - 3k\alpha^2 r^2, \quad (23)$$

from the energy in the field of Reissner-Nordstrom black hole without quintessence [4]. It may be pointed that due to the presence of quintessence the energy of black hole varies in the radial direction and has at most quadratic dependence on  $r$ . We now investigate the variation in the energy of such a black hole. Notice that the presence of the quintessence field will enhance the energy content of this black hole spacetime if  $E_Q$  in (23), is positive. This possibility can be ruled out easily by studying the graph of function  $E_Q$ . The function increases for all values of  $r$  in the range

$$r < \frac{5k - 2}{3k\alpha}. \quad (24)$$

Since  $0 < k \leq 1/4$  and  $\alpha > 0$ , therefore the quantity on the right hand side of (24) is always negative. This leads to a contradiction because  $r$  denotes radial distance and cannot be negative. Thus the enhancement of the energy content is not possible of the Reissner-Nordstrom black hole surrounded by quintessence, due to the presence of the quintessence field. The function  $E_Q$ , attains a maximum value at

$$r = \frac{5k - 2}{6k\alpha} < 0 \quad \left( \text{for all } 0 < k \leq \frac{1}{4} \right) \quad (25)$$

and then decreases with a negative slope

$$E_Q' = \alpha(5k - 2) - 6k\alpha^2 r < 0, \quad (26)$$

for all values of  $r$  in the range

$$\frac{5k-2}{6k\alpha} < r < \infty . \quad (27)$$

Therefore we conclude that in the above region the contribution due to quintessence term  $E_Q$ , is to decrease the energy in the spacetime of this black hole. This supports the idea of mass reduction of black holes by the accretion of dark energy on black holes [25, 26, 27].

Finally we find the value of  $r$  at which the term due to quintessence balances the energy content in the absence of this term. This can be done by equating the quintessence term  $E_Q$  to the negative of energy term  $(1-2k)$ , that corresponds to the energy without quintessence. We get two values of  $r$ , namely

$$r_{(\pm)} = \frac{(5k-2) \pm \sqrt{4-8k+k^2}}{6k\alpha} . \quad (28)$$

The discriminant is positive for  $0 < k \leq 1/4$  which gives the first root  $r_{(+)}$  as a positive quantity, i.e.,  $r_{(+)} > 0$ . On the other hand  $r_{(-)} < 0$ , which is not a feasible solution. Therefore, at the radial distance  $r = r_{(+)}$ , the effect of quintessence balances the energy content of the black hole without quintessence. While for  $r > r_{(+)}$ , the effect due to the quintessence term goes on decreasing. Figure 1 illustrates the behavior of  $E_Q$ , for different values of  $r$ .

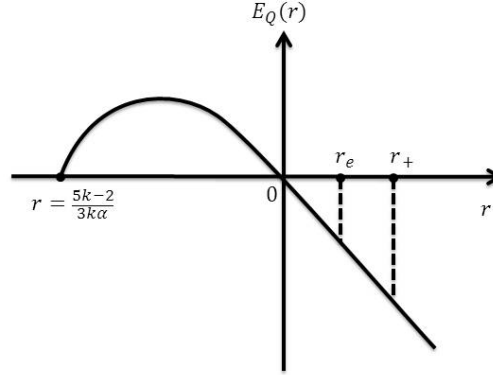


Figure – 1: Behavior of energy due to quintessence for  $0 < \alpha < 1$ . In this case, the radial distance  $r_+$  is outside the horizon  $r_e$  of black hole.



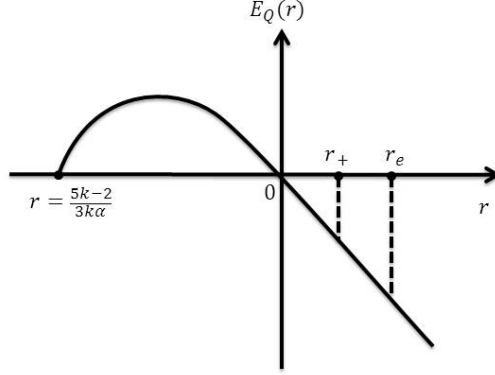


Figure – 2: Behavior of energy due to quintessence for  $\alpha \geq 1$ . In this case, the radial distance  $r_+$  is inside the horizon  $r_e$  of black hole.

It would be interesting to locate the point  $r_{(+)}$ , and check if it lies inside or outside the event horizon of the black hole. A detailed discussion on the event horizons of the Reissner-Nordstrom black hole surrounded by quintessence is given in [24]. We consider the case in which Reissner-Nordstrom black hole with quintessence has a single event horizon ( $r_e$ ), that correspond to a single real root of  $f(r)$ , in the metric. In this case equation for event horizon  $f(r) = 0$ , is a cubic equation in  $r$  and has three roots in general, one in the real and two on the complex plane. Since the complex roots leads to naked singularities therefore violate Roger Penrose's cosmic censorship hypothesis. Thus, the only possible real root correspond to a real singularity with a single event horizon. By taking suitable values of  $Q$  and  $M$ , satisfying  $M^2 \geq Q^2$ , it can easily be verified that (1) if  $0 < \alpha < 1$ , then the point  $r = r_+$  lies outside the event horizon and (2) for  $\alpha \geq 1$ , the point  $r = r_+$  lies inside the event horizon. Therefore we obtain critical bounds on the parameter  $\alpha$ , in the Reissner-Nordstrom black hole surrounded by a quintessence field. The case  $0 < \alpha < 1$ , is important because there exists a point outside the event horizon of black hole with quintessence where the energy of this spacetime vanishes completely. The other case  $\alpha \geq 1$ , brings  $r_+$  inside the event horizon of the black hole (see Figure 2) where fundamental laws of Physics fail to exist and nothing can be said about it.

## 5. Summary

In this paper we have investigated the energy content of the Reissner-Nordstrom black hole surrounded by quintessence field, by using second-order approximate Lie symmetries. For this purpose we have considered mass  $M$  and charge  $Q$  of the black hole as small parameter  $\epsilon$ . To determine the second-order approximate symmetries of the second-order perturbed geodesic equations we have first studied the exact (when  $\epsilon = 0$ ) and the first-order approximate (when second and higher powers of  $\epsilon$  are neglected) symmetries. In the exact case we have found the six dimensional Lie algebra with generators given in (19). In the first-order and second-order approximate cases we have no non-trivial approximate Lie symmetry generator. We have recovered the six exact symmetry generators as trivial first-order and second-order approximate symmetry generators.

From the application of the definition of the second-order approximate Lie symmetries we have obtained an energy rescaling factor (22) for the Reissner-Nordstrom black hole surrounded by quintessence. This energy rescaling factor depends on the ratio of the charge  $Q$  and mass  $M$  of the black hole and relates to the electromagnetic self-energy to the gravitational self-energy. These two parameters  $M$  and  $Q$  of the black hole appears quadratically in the energy rescaling factor (22). Further this rescaling factor for the Reissner-Nordstrom black hole with quintessence term also involve  $r$  and  $\alpha$ . The comparison of this rescaling factor with that for the Reissner-Nordstrom black hole [4], was given in Section 3. By taking  $\alpha = 0$  in (22), one can recover the energy rescaling factor for the Reissner-Nordstrom black hole [4]. The energy rescaling factor for the Reissner-Nordstrom black hole [4], do not depends on the coordinate  $r$ , while the factor (22) obtained here for the black hole in the presence of the quintessence matter involves  $r$ , which appears as a multiple of  $\alpha$  and does not appear separately. Taking  $\alpha = 0$ , the  $r$  dependent terms will also disappear from the rescaling factor (22). Here it is observed that the presence of the quintessence field may *reduce* the energy in the spacetime field of the Reissner-Nordstrom black hole surrounded by quintessence. Besides, we have obtained critical bounds on the value of the normalization factor  $\alpha$  i.e it lies between 0 and 1. It is also seen that outside the event horizon of the Reissner-Nordstrom black hole surrounded by quintessence, there exist a point at which the effect due to the quintessence term balances by the energy term of the black hole without the quintessence. The effect of quintessence then decreases beyond that point.

## Acknowledgments

IH would like to thank TWAS-UNESCO for awarding Associateship at Kavli Institute for Theoretical Physics, Chinese Academy of Sciences, Beijing, China where some part of the work was completed.

## References

- [1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, (W.H. Freeman, San Francisco, 1973).
- [2] L. B. Szabados, Living Rev. Relativ. **7**, 4 (2004).
- [3] M. Sharif, Nuovo Cimento B **118**, 669 (2003).
- [4] I. Hussain, F. M. Mahomed and A. Qadir, SIGMA **3**, 115 (2007).
- [5] I. Hussain, Definition of Energy in General Relativity via Approximate Symmetries, (LAP Lambert Academic Publishing, 2012).
- [6] N. H. Ibragimov, Elementary Lie group Analysis and Ordinary Differential Equations, (Wiley, Chichester, 1999).
- [7] I. Hussain, Phys. Scripta, **83**, 055002 (2011).
- [8] M. Sharif, and S. Waheed, Can. J. Phys. **88**, 833 (2010).
- [9] M. Sharif, and S. Waheed, Phys. Scr. **83**, 015014 (2011).
- [10] I. Hussain, J. Koer. Phys. Soc. (to appear).
- [11] I. Hussain, F. M. Mahomed, and A. Qadir, Phys. Rev. D **79** 125014 (2009).
- [12] I. Hussain, and A. Qadir, Nuovo Cimento B **122**, 593 (2007).
- [13] I. Hussain, F. M. Mahomed, and A. Qadir, Gen. Relativ. Grav. **41**, 2399 (2009).
- [14] I. Hussain, Gen. Relativ. Grav. **43** 1037 (2011).
- [15] S. Perlmutter et al., Astrophys. J. **517**, 565 (1999).
- [16] P. de Bernardis et al., Nature, **404**, 955 (2000).
- [17] D. N. Spergel et al., Astrophys. J. Suppl. **148**, 175 (2003).
- [18] W. Miranda, S. Carneiro, C. Pigozzo, JCAP 1407, 043 (2014).
- [19] I. Zlatev, L. -M. Wang and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
- [20] V V Kiselev, Class. Quantum Grav **20**, 1187 (2003).
- [21] N. H. Ibragimov, A. H. Kara and F. M. Mahomed, Lie-Bäcklund and Noether Symmetries with Applications, Nonlinear Dynamics **15**, 115 (1998).
- [22] E. Noether, Math. Phys.Kl. 2 (1918) 235. (English translation in transport theory and Statistical Physics 1 (1971)) 186.

- [23] L. V. Ovsiannikov, Group Analysis of Differential Equations, (New York: Academic Press, 1980).
- [24] S. Fernando, Gen. Rel. Grav. **45**, 2053 (2013).
- [25] E. Babichev, V. Dokuchaev, and Yu. Eroshenko, Phys. Rev. Lett. **93**, 021102 (2004).
- [26] M. Jamil, Eur. Phys. J. C **62**, 609 (2009).
- [27] M. Jamil and A. Qadir, Gen. Rel. Grav. **43**, 1069 (2011).